Instantons For Black Hole Pair Production*

Paul M. Branoff† and Dieter R. Brill‡

The University of Maryland
Department of Physics
College Park, Maryland 20740

Abstract

We address the issue of constructing continuous instantons representing the pair creation of black holes in a cosmological context. The recent attempt at constructing such solutions using virtual domain walls is reviewed first. We then explore the existence of continous instantons in higher curavature gravity theories where the Lagrangian is polynomial in the Ricci scalar. Lastly, we study continous instanton solutions of ordinary gravity coupled to the Narlikar C-field. For each theory, we first consider the case of finding continuous instanton solutions which represent the "near-annihilation" of a de Sitter universe and its subsequent recreation. In situations where these solutions exist, we then ask whether solutions can be found that represent the "near-annihilation" of a de Sitter spacetime and the subsequent creation of a pair of Schwarzschild-de Sitter black holes.

1. Introduction

Of Jayant Narlikar's many important contributions to astrophysics and cosmology, none is more creative and imaginative than the idea, developed with Fred Hoyle, that particles may be created as the universe expands. Stated long before quantum effects of gravity could be treated, this proposal has new meaning today. Methods are now available to analyze quantum particle production in dynamic spacetimes, and even black hole creation can be understood semiclassically as a tunneling process. The latter process is the main subject of this paper.

Although a complete theory of quantum gravity does not yet exist, examples of gravitational tunneling have been studied for a number of years, including such processes as pair creation of black holes and vacuum decay of domain walls. In each case the treatment is based on an instanton (solution of the Euclidean field equations) that connects the states between which tunneling is taking place. However, there are some nucleation processes of interest where the standard instanton method is not effective, for example because no solutions exist to the Euclidean Einstein equations that smoothly connect the spacelike sections representing the initial and final states of the tunneling process. It is therefore an interesting challenge to adapt the "bounce" method, most suitable for vacuum decay calculations,

*To be published in a festschrift for J. V. Narlikar, Kluwer Academic Publishers, 1999.
†email: branoff@glue.umd.edu
‡email: brill@physics.umd.edu

to deal with non-static initial states and background fields such as a positive cosmological constant or domain walls typically present when particle-like states are created.

A positive cosmological constant (and other strong gravitational sources, such as a positive energy density domain wall) acts to increase the separation of timelike geodesics. It is therefore expected to "pull particles out of the vacuum" by favoring creation of pairs over their annihilation. The analogous creation of black hole pairs in de Sitter space can be treated in WKB approximation by the "no boundary" realization of quantum cosmology [1]. The first (and usually only) step in such a treatment consists of finding a solution of the Euclidean field equations containing the initial state (pure de Sitter universe) and the final state (Schwarzschild-de Sitter space) as totally geodesic boundaries. Such a solution exists only if we accept it in two disconnected pieces. If the cosmological constant is large enough one then obtains an appreciable probability of creating in each Hubble volume a pair of black holes comparable to the volume's size; if these break up into smaller ones (see, for example, Gregory and Laflamme [2]) one has, within pure gravity, a model of continuous creation not too far removed in spirit from that of Hoyle and Narlikar.

This model is, however, not fully satisfactory in several respects. For example, it is not clear how to calculate the "prefactor" of the exponential in the transition probability, which would define the dimensionful rate of the process. When it can be calculated from the fluctuations about the instanton [3], a "negative mode" is necessary for a non-vanishing rate. But this negative mode would have to connect the two parts of the instanton, and therefore cannot be treated as a small perturbation. A discontinuous instanton is of course also conceptually unsatisfactory, because the usual composition rules assume that histories are continuous.

Each of the two parts of the disconnected instanton has the universe's volume reaching zero. By forbidding arbitrarily small volumes one can connect the two parts. The exploration of modifications of Einstein gravity in which this is possible is still in its infancy. For example, Bousso and Chamblin [4] have used virtual domain walls to construct interpolating instantons. A similar technique using 'pseudomanifolds' has also been used to construct such solutions [5].

Modifications of Einstein's theory that have been proposed in other contexts may also give continuous instantons, if the change from Einstein's theory becomes important at small volumes. For this reason it is natural to consider higher curvature gravity theories.

Another promising modification of Einstein gravity is Narlikar's C-field [6]. This field can describe reasonable energetics of particle production in a context where quantum mechanics plays no essential role, and it is therefore interesting to explore, as we will below, whether it can also solve the disconnectedness problem in the instanton treatment.

So we ask whether these modifications of the Einstein-Hilbert action allow continuous paths from an initial to final cosmological state when calculating amplitudes for cosmological black hole production in the context of closed universes. We will outline a modified version of the calculation of Bousso and Chamblin concerning the use of virtual domain walls in constructing interpolating instantons. We next discuss the existence of continuous instantons in higher curvature gravity theories whose Lagrangians are nonlinear in the Ricci scalar.

Finally, we consider the case of general relativity with a cosmological constant and a Narlikar C-field.

2. Gravitational Tunneling

Processes such as black hole pair creation can be analyzed semi-classically through the use of instanton methods. One can think of such a process as a tunneling phenomenon. The initial state consists of a universe with some background metric and no black holes, and the final state consists of a universe with two black holes supplementing the background metric. Classical dynamics is prevented from connecting the two states by a generalized potential barrier. The quantum process can "penetrate" the barrier with some probability, and the same barrier makes it improbable for the final state, once created, to "annihilate" back to the initial state. In problems that can be treated by instantons, the non-classical transition from initial to final state can be described approximately as an excursion in imaginary time. A solution that goes from the initial state to the final state and back again is called a bounce solution; an instanton is a solution which goes from the initial state to the final state, i.e., half a bounce. In the WKB interpretation the excursion into imaginary time simply signifies an exponentially decreasing wavefunction that is large only near configurations contained in the instanton. In the sum over histories interpretation the instanton is a saddle point by means of which the propagator is to be be evaluated.

The exponential of the instanton's classical Euclidean action is the dominant factor in the transition probability, provided it is normalized so that the action vanishes when there is no transition. That is, we are really comparing two instantons, one corresponding to the background alone in which initial and final states are the same, and the instanton of the bounce, in which they are different. If the initial state is static, it is typically approached asymptotically by the bounce, and therefore the normalization of the action can be achieved by a suitable surface term. If the initial state is only momentarily static, as in the case of the de Sitter universe, we must find the two instantons explicitly and evaluate their actions. In the context of the disconnected instanton the background instanton corresponds to two disconnected halves of a 4-sphere: a de Sitter space fluctuating into nothing and back again. A first test whether a modification of Einstein's theory can have connected instantons is therefore to see whether the background instanton can be connected (Fig. 1).

The rate of processes like black hole pair creation is calculated by subtracting from the action of the bounce, $I^{\rm bc}$, the action corresponding to the background state, $I^{\rm bg}$. The pair creation rate is then given as

$$\Gamma = A \exp\left[-(I^{\rm bc} - I^{\rm bg})\right], \tag{1}$$

where A is a prefactor, which is typically neglected in most calculations because it involves fluctuations about the classical instantons that are difficult to calculate. Without this dimensionful prefactor one can find the relative transitions to different final states, but the actual the number of transitions per spacetime volume to a given final state can only be estimated, for example as 1/(instanton four-volume) for finite volume instantons.

The connected background instanton as described above is closely related to a Euclidean

wormhole, or birth of a baby universe [7]: if the two parts are connected across a totally geodesic 3-surface, we can, according to the usual rules, join a Lorentzian space-time at that surface, passing back to real time. An instanton with this surface as the final state would then describe the fluctuation of a large universe into a small one, with probability comparable to that of the creation of a black hole pair. Thus whatever process provides a connected instanton is likely to lead not only to the pair creation but also to formation of a baby universe. (In section 5 we will see how the latter can be avoided)

An instanton calculation has been used by a number of authors to find the pair creation of black holes on various backgrounds (see, for example, Garfinkle *et al.* [8]). The instantons involved a continuous interpolation between an initial state without black holes and a final state with a pair of black holes. By contrast, in cosmological scenarios where the universe closes but Lorentzian geodesics diverge, as in the presence of a positive cosmological constant or a domain wall, there are Lorentzian solutions to Einstein's equations with and without black holes (such as de Sitter and Schwarzschild-de Sitter spacetimes, respectively), but there are no Euclidean solutions that connect the spacelike sections of these geometries [4]. (For the related case of baby universe creation the absence of such solutions is understood, for it is necessary that the Ricci tensor have at least one negative eigenvalue [9].)

The No-Boundary Proposal of Hartle and Hawking [1] can be modified to provide answers in these cases. The original proposal was designed to eliminate the initial and final singularities of cosmological models by obtaining the universe as a sum of regular histories, which may include intervals of imaginary time. One can think of the Euclidean sector of the dominant history as an instanton that mediates the creation of a (typically totally geodesic) Lorentzian section from nothing. By calculating the action corresponding to these instantons, one can calculate the wave function for this type of universe, i.e.,

$$\Psi(\mathcal{G}) = e^{-I_{\text{inst}}(\mathcal{G})} \tag{2}$$

where $I_{\text{inst}}(\mathcal{G}) = \frac{1}{2} I^{\text{bc}}$ is the action corresponding to a saddlepoint solution of the Euclidean Einstein equations whose only boundary is the 3-dimensional geometry $\mathcal{G}$. The probability measure associated with this universe is then given by

$$P = \Psi^* \Psi = e^{-2 I_{\text{inst}}} \tag{3}$$

To relate the probability measure to the pair creation rate of black holes given in Eq. (1) one writes

$$\Gamma = \frac{P_{\text{bh}}}{P_{\text{bg}}} = \exp[-(2 I_{\text{inst}}^{\text{bh}} - 2 I_{\text{inst}}^{\text{bg}})] \tag{4}$$

so the ratio of the probability of a universe with black holes to the probability of a background universe without black holes is taken to be also the rate at which an initial cosmological state can decay into a final cosmological state, that is, the pair creation rate. In the latter sense the two disconnected instantons together describe the tunneling process.

Although this formalism allows one to calculate, in principle, the rates of nucleation processes, there is no well-justified reason why Eq. (4) should be identified with this quantity. The straightforward interpretation of the instanton concerns the probability for one universe

to annihilate to nothing and for a second universe to be nucleated from nothing. This second universe can either contain a pair of black holes, or it can be identical to the initial universe, but it retains no "memory" of the initial state. It would clearly be preferable to have a continuous interpolation between the initial and final states. (This would allow degrees of freedom that interact only weakly with the dynamics of gravity to act as a memory that survives the pair creation.) In the following sections we will consider several ways in which this continuity of spacetime can be achieved, the first of which involves matter fields that can form virtual domain walls.

3. Continuous Instantons via Virtual Domain Walls

In this section, we will consider the method by which the authors of [4] use virtual domain walls to construct continuous paths between two otherwise disconnected instantons. They illustrated the method for the nucleation of magnetically charged Reissner-Nordström black holes in the presence of a domain wall. We will confine attention to nucleation of uncharged black holes in a universe with a cosmological constant. The initial state is the de Sitter universe and the final state is the extremal form of a Schwarzschild-de Sitter universe known as the Nariai universe [10], which is dictated by the requirement that the Euclidean solution be non-singular. To understand virtual domain walls we will need some elementary properties of real domain walls. These have been discussed extensively in [4, 11, 12, 13, 14, 15].

3.1 Brief Overview of Domain Walls

A vacuum domain wall is a $(D-2)$-dimensional topological defect in a D-dimensional spacetime that forms as a result of a field ϕ undergoing the spontaneous breaking of a discrete symmetry. If we let $\mathcal{M}$ denote the manifold of vacuum expectation values of the field ϕ, then a necessary condition for a domain wall to form is that the vacuum manifold is not connected ($\pi_0(\mathcal{M}) \neq 0$). An example of a potential energy function $U(\phi)$ of the field ϕ giving rise to domain walls is the double-well potential.

Throughout this section, we will be dealing with a domain wall in the "thin-wall" approximation, which means that the thickness of the domain wall is negligible compared to its other dimensions, and it is homogeneous and isotropic in its two spacelike dimensions, so that the spatial section of the wall can be treated as planar, and the spacetime geometry as reflection symmetric with respect to the wall.

The action of a real scalar field ϕ, interacting with gravity, that may form a domain wall is given by

$$I_{\rm dw} = \int d^4x\sqrt{-g}\,\left[L_{\rm mat} + \frac{R-2\Lambda}{16\pi}\right] \tag{5}$$

with matter Lagrangian

$$L_{\rm mat} = -\frac{1}{2}\,g^{\mu\nu}\,\partial_\mu\phi\partial_\nu\phi - U(\phi) \tag{6}$$

and stress-energy tensor

$$T_{\mu\nu} = \partial_\mu\phi\,\partial_\nu\phi - g_{\mu\nu}\left[\frac{1}{2}\,g^{\alpha\beta}\partial_\alpha\phi\,\partial_\beta\phi + U(\phi)\right]\,. \tag{7}$$

Here $U(\phi)$ is a potential function with two degenerate minima ϕ_- and ϕ_+, at which $U = 0$; g is the determinant of the 4-metric $g_{\mu\nu}$; and R is the Ricci scalar. (We have neglected boundary terms in the action since the instantons we will be considering are compact and have no boundary.)

The trace of the Einstein equations (resulting from the variation of $I_{\rm dw}$ with respect to $g_{\mu\nu}$) gives

$$\frac{R-4\Lambda}{8\pi} = g^{\mu\nu}\,\partial_\mu\phi\,\partial_\nu\phi + 4U(\phi)\,, \tag{8}$$

which can be used to simplify the action (5) when evaluated on a solution:

$$I_{\rm dw} = \int \left[U(\phi) + \frac{\Lambda}{8\pi}\right]\sqrt{-g}\,d^4x\,. \tag{9}$$

The ϕ-field is essentially constant away from a domain wall, with $\phi = \phi_-$ on one side and $\phi = \phi_+$ on the other. In Gaussian normal coordinates (ζ^i, z) with the wall at $z = 0$, ϕ depends only on z, and the field equation for ϕ implies that T_{zz} of Eq. (7) is negligible. The rest of the components of the stress-energy tensor differ from zero only near the wall, where ϕ changes rapidly from ϕ_- to ϕ_+:

$$T^\mu_\nu = \sigma\delta(z)\mathrm{diag}(1,1,1,0) \tag{10}$$

where σ can be related via the ϕ-field equation to the ϕ-potential alone,

$$\sigma = \int 2U(\phi(z))dz\,. \tag{11}$$

Thus σ is the surface energy density of the wall. For such surface distributions the Israel matching condition imply that the intrinsic geometry h_{ij} of the domain wall is continuous, and that the extrinsic curvature jumps according to

$$K^+_{ij} - K^-_{ij} = 4\pi\sigma h_{ij}\,. \tag{12}$$

Here the normal with respect to which K_{ij} is defined points from the + side of the surface to the − side. Outside the wall we have the sourceless Einstein equations.

3.2 Joining Instantons by Domain Walls

The jump (12) in extrinsic curvature across a domain wall can be used to join the two parts of a disconnected instanton (Fig. 1) by "surgery": We remove a small 4-ball of radius η from each instanton. Their two 3-surface boundaries have the same intrinsic geometry, and their extrinsic curvatures are proportional to the surface metric. They can therefore be joined together in such a way as to satisfy the Israel matching conditions, Eq. (12), thereby inserting a domain wall.

However, the surface energy density $\tilde{\sigma}$ of the domain wall used to join the instanton must be negative: As we approach the domain wall from the initial state, heading towards annihilation, successive 3-spheres are shrinking, $K^+_{ij} < 0$. After we pass through the domain wall, successive 3-spheres are expanding, $K^-_{ij} > 0$. Because of the negative energy density

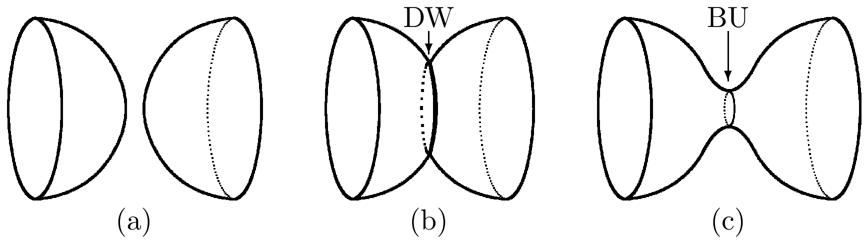

Figure 1: Two-dimensional analog of de Sitter instanton. Imaginary time runs horizontally. Because no significant change can be shown in two dimensions, this is a "background" instanton with identical initial and final states. (a) The disconnected instanton. (b) "Yoyo" instanton connected by domain wall (heavy curve labeled DW). (c) Instanton connected by a "virtual baby universe" (BU).

the authors of [4] call this a virtual domain wall, but it is not virtual in the sense that it corresponds to a Euclidean solution of the equations of section 3.1, for the σ of Eq. (11) remains positive when passing to imaginary time. Within this scheme the only way to achieve a "yoyo" instanton as a saddle point of the Euclidean action is to have a scalar field with a negative energy also in the real domain, that is, a Lagrangian with the opposite sign as that of Eq. (6). As we will see in section 5, in that case a plain scalar field, without the domain-wall-forming potential $U(\phi)$, will do as well and is preferable.

By how much does the Euclidean action change when we introduce a domain wall whose radius η is small compared to the radius $\sqrt{3/\Lambda}$ of the instanton itself? The extrinsic curvature of the connecting 3-sphere is then nearly the same as what it would be in flat space, $K_{ij} = h_{ij}/\eta$, and the jump in curvature is twice that; hence the size of the domain wall is determined from Eq. (12),

$$\eta = -\frac{1}{2\pi\tilde{\sigma}}\,. \tag{13}$$

The Euclidean version of Eq. (9) is

$$I_{\rm dw} = -\int \left[U(\phi) + \frac{\Lambda}{8\pi}\right] \sqrt{g}\, d^4x\,. \tag{14}$$

We have taken a 4-ball with scalar curvature $R = 4\Lambda$ away from each part of the original instanton, for a total change in action by $\pi\Lambda\eta^4/16$; this is small compared to that due to the added domain wall with action given by Eqs. (14) and (11), $I_{\rm dw} = -\pi^2\tilde{\sigma}\eta^3 = \frac{1}{2}\pi\eta^2 = \frac{1}{64\pi} \times \frac{\eta^2}{3/\Lambda} \times \frac{96\pi^2}{\Lambda}$, which is small compared to the total action $96\pi^2/\Lambda$. Thus the Euclidean action increases when we add the domain wall, and the connected instanton therefore has a relatively smaller probability measure (although the difference is small compared to the total action), and the disconnected instanton will dominate. If the path integral is extended over continuous histories only, the domain wall provides the only saddle point, with action very close to what the discontinuous history would have given, thus justifying the calculation using the discontinuous history alone. But in that case a path integral without a domain-wall-forming scalar field gives a very similar value of the action, as shown in [4].

Introducing this scalar field may therefore be considered a high price to pay for gaining a saddle point, particularly because it entails other, less desirable processes. For example,

the "center" $z = 0$ of the domain wall is totally geodesic with $\partial\phi/\partial z = 0$, that is, a possible place to revert from imaginary time back to real time. This corresponds to the formation of a baby universe of size comparable to η and smaller Euclidean action than that for the black hole formation.

If a field exists that can form small domain walls, any two instanton parts can be connected by such surgery across one or several small 3-spheres, with a change in action as estimated above for each; the dominant history will have the fewest connections.

Finally, recall that the periodicity in imaginary time of each part of the disconnected instanton is well defined by the requirement that conical singularities should be absent from each part. If the parts are connected where there would otherwise be a conical singularity, one such requirement is eliminated. Thus there are connected instantons for which the final state is not Nariai but Schwarzschild-de Sitter geometry with black hole and cosmological horizons of unequal size.

4. Continuous Instantons in Higher Curvature Theories

Higher curvature theories have a long history and have been proposed in several different contexts. For example, they arise naturally in theories describing gravity by an effective action [16, 17].

In this section we will explore whether higher order theories can pass the "first test" of Section 2, namely whether there is a continuous instanton describing the annihilation and rebirth of de Sitter space (generalized to these theories). Adding higher order terms to the action does not, however, immediately eliminate disconnected instantons; for example, de Sitter space (that is, a spacetime of constant curvature) is a solution of many higher-order theories. In fact, if the universe without and with black holes can originate by tunneling from nothing, a disconnected instanton will also exist. Therefore connected instantons may again co-exist with the de Sitter-like, disconnected instantons.

The Euclidean action we will be considering has the form

$$I = -\frac{1}{16\pi}\int f(R)\sqrt{g}\, d^4x \tag{15}$$

where

$$f(R) = R - 2\Lambda + \alpha R^2 + \gamma R^3 + \cdots\,, \tag{16}$$

R is the Ricci scalar, Λ is the cosmological constant, and α, γ, etc., are coupling constants whose value we leave unspecified for the moment. The metric has the Euclidean Robertson-Walker form appropriate to three-dimensional space slices of constant positive curvature:[1]

$$ds^2 = N^2(\tau)d\tau^2 + a^2(\tau)d\Omega_3^2\,. \tag{17}$$

Here τ is imaginary time determined from the analytic continuation $t \to i\tau$, N is the lapse function, a is the universe radius and $d\Omega_3^2$ is the metric on the unit three-sphere. Having

[1]The cases of zero or negative curvature present additional normalization problems because the naive Euclidean action would be infinite. Therefore we confine attention to the positive curvature case.

the metric depend on N and a allows us to obtain all the independent Einstein equations by varying only these functions in the action (15): variation with respect to a gives us the one independent spacelike time development equation, and variation with respect to N yields the timelike constraint equation, as in ordinary Einstein theory. A further variation that is easily performed is a *conformal* change of the metric, giving the trace of the field equations, which is not independent of the other equations but involves only the function $f(R)$:

$$\begin{aligned}\frac{\partial}{\partial a}(fNa^3) - \frac{d}{d\tau}\left[\frac{\partial}{\partial \dot{a}}(fNa^3)\right] + \frac{d^2}{d\tau^2}\left[\frac{\partial}{\partial \ddot{a}}(fNa^3)\right] &= 0 \qquad (18)\\ a^3 f + \frac{\partial f}{\partial N}Na^3 - \frac{d}{d\tau}\left[\frac{\partial}{\partial \dot{N}}(fNa^3)\right] &= 0 \qquad (19)\\ 2Rf' + 6\nabla^2 f' - 4f &= 0 \qquad (20)\end{aligned}$$

where

$$\nabla^2 = \frac{d^2}{d\tau^2} + \frac{3\dot{a}}{a}\frac{d}{d\tau}, \qquad (21)$$

a dot denotes $d/d\tau$, and a prime denotes d/dR.

Equation (18) is a fourth order ordinary differential equation, and Eq. (19) is a third order first integral of this equation. The trace equation (20) shows that we can regard R as an independent variable, satisfying a second order equation. In this view Eq. (20) replaces Eq. (18) (to which it is equivalent), and a also satisfies a second-order differential equation, namely its definition in terms of R,

$$R = -6\left(\frac{\ddot{a}}{aN^2} - \frac{\dot{a}\dot{N}}{aN^3} + \frac{\dot{a}^2}{a^2N^2} - \frac{1}{a^2}\right). \qquad (22)$$

In addition we still have the constraint, Eq. (19), a first order relation between a and R.

A general Hamiltonian analysis (c.f. [18] and references therein), not confined to the symmetry of Eq. (17), bears out the idea that, as a second-order field theory, this is Einstein theory coupled to a non-standard scalar field [19]. For example, for a Lagrangian quadratic in the Ricci scalar with no cosmological constant, the relationship between R and the non-standard scalar field ϕ is given by [20]

$$\phi = \sqrt{\frac{3}{4\pi}}\alpha R \qquad (23)$$

where the ϕ-field has the standard stress energy tensor multiplied by $(1 + 4(\pi/3)^{1/2}\phi)^{-2}$.

Can this effective scalar field form a domain wall in four dimensions? If the coefficients up to γ in Eq. (16) are non-zero, then the "force term" $Rf' - 2f$ occurring in the trace equation (20) vanishes at three equilibrium states for R, where $\nabla^2 f'(R) = 0$, approximately at $R = \pm 1/\sqrt{\gamma}$ and at $R = 4\Lambda$, for small γ. But in order to have a macroscopic universe on either side of the wall we need $R = 4\Lambda$ on either side, so the usual wall formation where the scalar field changes from one equilibrium to another is unsuitable in this case. A solution of the bounce type may appear possible, since the equilibrium at $R = 4\Lambda$ is unstable. At the

turning point the time-dependent R would then have to "overshoot" the stable equilibrium near $R = -1/\sqrt{\gamma}$. A negative R is required there so that the universe radius can turn around at the same moment. This synchronization, if possible at all (numerical calculations have failed to reveal it to us; see however ref. [21]), would require fine tuning that does not appear natural in this context. Furthermore, if we had a bounce for both a and R, half of it would be an instanton describing the formation of a baby universe of size $\sim \gamma^{-1/4}$, which would then continue to collapse classically, and this process would be exponentially more probably than the black hole formation. For these reasons the effective scalar field that derives from higher curvature Lagrangians of the form (15) does not appear promising for connected instantons.

We therefore consider solutions to Eqs (18) - (20) when R is constant, $R = R_0$. To allow a continuous transition to imaginary time at $\tau = 0$ we make the usual ansatz that all odd time derivatives of $a(\tau)$ vanish at $\tau = 0$. With the choice $N = 1$, the above equations at $\tau = 0$ take the form

$$\begin{aligned} R_0 f' - 2f &= 0 && (24)\\ a_0 f + 6\ddot{a}_0 f' &= 0 && (25)\\ a_0^2 f - 2f'(a_0\ddot{a}_0 - 2) &= 0\,. && (26) \end{aligned}$$

Eliminating f from these equations we get the condition

$$(a_0^2 R_0 - 12) f' = 0\,. \tag{27}$$

Thus, we have two classes of solutions. The first class is described by the condition $R_0 = 12/a_0^2$. The second class is described by the condition $f = f' = 0$.

If $R = R_0 = 12/a_0^2$ and $\dot{a}_0 = 0$ then the unique regular solution of Eq. (22) is a de Sitter-like solution, $a(\tau) = a_0 \cos(\tau/a_0)$, leading only to the disconnected instanton.

The second condition indeed allows periodic, non-collapsing solutions with any amplitude A of the form

$$a^2 = \frac{6}{R_0} + A\cos\sqrt{\frac{R_0}{3}}\tau \qquad \text{where} \qquad f(R_0) = 0 = f'(R_0)\,. \tag{28}$$

If we want this R_0 to be close to that of the de Sitter universe, $R_0 = 4\Lambda$, then at least one of the higher-order coefficients (α, γ ...) in $f(R)$ of Eq. (16) has to be large and rather fine tuned. Furthermore, because the action for all of these solutions vanishes, we should integrate over all values of A, which includes some disconnected instantons, so this problem is not really avoided by these solutions. (They appear pathological also in other ways, for example they would allow production of baby universes of any radius. They would also tend to be unstable in the Lorentzian sector, although this can be confined to the largest scale by judiciously choosing $f(R) = R - 2\Lambda$ except near $R_0 \sim 2\Lambda$.)

5. Continuous Instantons in C-field Theory

Except for boundary terms, which describe classical matter creation and which we neglect in the present context, the C-field Lagrangian is similar to the usual scalar field Lagrangian without self-interaction (Eq. (6) with $U = 0$), but with the important difference

that the coupling constant $-f$ of the C-field has the opposite sign from the usual one [6]. Thus the total action of gravity with cosmological constant and C-field has the form, for Lorentzian geometries

$$I_C = \int d^4x\sqrt{-g}\,\left[\frac{1}{2}f\,g^{\mu\nu}\,\partial_\mu C\partial_\nu C + \frac{R-2\Lambda}{16\pi}\right]. \tag{29}$$

(We have not included ordinary matter fields here because we are confining attention to pair production of black holes as purely geometrical objects.)

5.1 Sourceless C-field in Lorentzian Cosmology

The field equations that follow from this action by varying C and $g_{\mu\nu}$ are, for the C-field:

$$\Box C = C^{\mu}_{\ ;\mu} = 0 \quad (\text{where } C_\mu = C_{,\mu}) \tag{30}$$

and for the geometry,

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = T^C_{\mu\nu} \quad \text{where} \quad T^C_{\mu\nu} = -f\left(C_\mu C_\nu - \frac{1}{2}g_{\mu\nu}C^\alpha C_\alpha\right). \tag{31}$$

The stress-energy tensor $T^C_{\mu\nu}$ gives a negative energy density (for $f > 0$). Narlikar [6] has given reasons why this violation of the energy condition is not an objection when the C-field is coupled to Einstein gravity of an expanding universe.

For Lorentzian cosmology we make a Robertson-Walker ansatz analogous to (17),

$$ds^2 = -N^2(t)dt^2 + a^2(t)d\Omega_3^2\,. \tag{32}$$

In agreement with the homogeneous nature of this geometry we assume that C is homogeneous in space and hence depends only on t. The field equations, derived by varying a, N, and C, and then setting $N = 1$, are

$$\begin{aligned} 2\frac{\ddot{a}}{a} + \frac{\dot{a}^2+1}{a^2} - \Lambda &= 4\pi f\dot{C}^2 & (33)\\ \dot{a}^2 + 1 - \frac{\Lambda}{3}a^2 &= -\frac{4\pi f}{3}\,\dot{C}^2 a^2 & (34)\\ \frac{d(a^3\dot{C})}{a^3\,dt} &= 0\,. & (35)\end{aligned}$$

The second equation, as usual, is a first integral of the first (time development) equation, and it implies the latter except for extraneous solutions $a =$ const. The third equation has the integral

$$\dot{C} = \frac{K}{a^3} \tag{36}$$

where K is a constant. By eliminating $\dot{C}$ we obtain an equation of the "conservation of energy" type for a:

$$\dot{a}^2 + V_{\text{eff}} = \dot{a}^2 - \frac{\Lambda}{3}a^2 + \frac{4\pi f K^2}{3a^4} = -1\,. \tag{37}$$

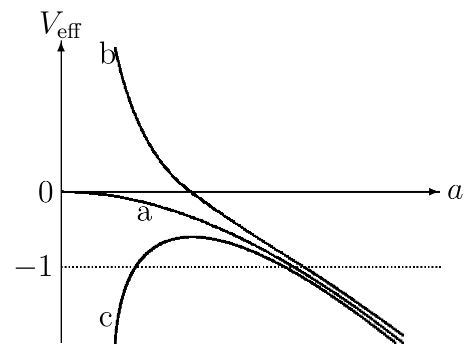

Figure 2: The effective potential for de Sitter-like universes: a universe with only cosmological constant (curve a), one with a real C-field (curve b), and one with a virtual C-field (curve c).

This is the usual de Sitter equation supplemented by a term in $1/a^4$, which is unimportant at late times when a is large and does not change the qualitative Lorentzian time development at any time (Fig. 2).

5.2 Sourceless C-field in Euclidean Cosmology

The effective potential in Eq. (37) increases monotonically as a decreases below the minimum classically allowed value. The corresponding Euclidean motion in such a potential therefore does not bounce; instead, a would continue to decrease and reach $a = 0$ in a finite Euclidean time. This is a geometrical singularity if $K \neq 0$ because, for example, it follows from Eqs. (33) and (36) that $R = 4\Lambda + 8\pi f(K^2/a^6)$.

However, a different potential is obtained if the motion of both a and C is continued to imaginary time,[2] thereby describing a virtual process that involves both of these variables, so that we take into account fluctuations in C as well as in a. Then the K of Eq. (36) becomes imaginary, $K = ik$, and the Euclidean "conservation of energy" equation becomes

$$-\dot{a}^2 - \frac{\Lambda}{3}a^2 - \frac{4\pi f k^2}{3a^4} = -1\,. \tag{38}$$

It is easily seen that, for sufficiently small k, this equation does have bounce solutions, with a turning point at $a \sim k^{1/2} f^{1/4}$ (Fig. 2). Thus the C-field theory passes the "first test": it has a continuous instanton describing a fluctuation with identical initial and final state. It is reasonable to suppose that the theory will also have continuous instantons describing the creation of a black hole pair, because for small k the turning point occurs at small a, so that two disconnected instantons can be joined by surgery similar to that of section 3.

It is essential that the fluctuation of the C-field be virtual, that is, that the coupling constant f have the opposite sign from the usual, positive energy density scalar field. If the

[2] We assume that this transition is the most probable; this would not be so if a transition were possible in the potential of Eq. (37). For example in penetrating radially a spherically symmetric potential barrier the most likely transition maintains the real angular momentum [22].

C-field were real, time could revert to real values at the minimum radius of the bounce and continue in a small, Lorentzian universe [23] that we have above described as a baby universe. This transition would be the most probable if allowed. By contrast, in the case of the virtual C-field this transition is not allowed, The reason is that at the moment of the bounce, the C-field's effective potential dominates. A return to real time (K changing from imaginary to real) would make a large change in $V_{\rm eff}$ of Eq. (37), violating this Lorentzian Hamiltonian constraint. A much smaller violation is involved at the first change to imaginary time, at large a. This can occur if the background is not exactly de Sitter-like, but contains some gravitational wave excitation that can supply the necessary small energy difference in the local region where the black hole will form. Thus the C-field makes a continuous instanton possible, but avoids forming a baby universe.[3]

5.3 Black Holes in C-field Cosmology

As a final step we exhibit as an endstate of the particle creation instanton an expanding universe in C-field theory of spatial topology $S^1 \times S^2$. This describes a universe with an extremal black hole pair in the same sense that the Nariai solution [10, 25] describes such a universe in Einstein's theory. The metric has the homogeneous form

$$ds^2 = -dt^2 + a^2(t)d\chi^2 + b^2(t)(d\theta^2 + \sin^2\theta d\phi^2) \tag{39}$$

where χ has periodicity appropriate to S^1, θ and ϕ are coordinates on S^2, and a and b are functions only of t. The C-field likewise is a function only of t and therefore obeys the conservation law analogous to (36),

$$\dot{C} = \frac{K}{ab^2}\,. \tag{40}$$

The field equations then take the form

$$\begin{aligned} G^t_t + \Lambda = -\frac{2\dot{a}\dot{b}}{ab} - \frac{\dot{b}^2+1}{b^2} + \Lambda &= \frac{4\pi f K^2}{a^2b^4} && (41)\\ G^\chi_\chi + \Lambda = -\frac{2\ddot{b}}{b} - \frac{\dot{b}^2+1}{b^2} + \Lambda &= -\frac{4\pi f K^2}{a^2b^4} && (42)\\ G^\theta_\theta + \Lambda = -\frac{\ddot{a}}{a} - \frac{\dot{a}\dot{b}}{ab} - \frac{\ddot{b}}{b} + \Lambda &= -\frac{4\pi f K^2}{a^2b^4}\,. && (43) \end{aligned}$$

If the universe volume expands similar to the Nariai solution, the effects the C-field will become negligible at late times. It is therefore reasonable to solve the field equations with the condition that the solution be asymptotic to the Nariai universe, $a(t) = (1/\sqrt{\Lambda})\cosh\sqrt{\Lambda}t$, $b(t) = 1/\sqrt{\Lambda}$. We also require a moment of time-symmetry (to enable the transition from imaginary time). The solution to first order in $\varepsilon = 4\pi f K^2\Lambda^{3/2}$ is

$$\begin{aligned} a(t) &= \frac{1}{\sqrt{\Lambda}}\cosh\sqrt{\Lambda}t - \frac{\varepsilon}{3}\ln(2\cosh\sqrt{\Lambda}t) - \frac{\varepsilon}{8}e^{-2\sqrt{\Lambda}t} + \cdots && (44)\\ b(t) &= \frac{1}{\sqrt{\Lambda}} + \frac{\varepsilon}{6}e^{-2\sqrt{\Lambda}t} + \cdots\,. && (45) \end{aligned}$$

[3] We also note that, as remarked in [24], a real change in C (if K were real) during the instanton could be interpreted as a change in the gravitational constant after the pair creation, which would be undesirable.

These functions do not differ much from those for the Nariai solution for any time t. However, the differences would become large in the continuation to imaginary time, as the volume decreases. In order to reach a minimum volume we again need an imaginary K (virtual C-field). This minimum volume, like all $t =$ const. surfaces, has topology $S^1 \times S^2$ and would therefore not fit directly on the minimum-a surface of a de Sitter-like metric, Eq. (17); a solution with less symmetry in both spaces would be needed to make the match.

6. Conclusions

In Einstein's theory of gravity with a cosmological constant, typical Euclidean solutions describe a universe originating from "nothing," or decaying into nothing, but there are no equally simple solutions corresponding to quantum processes, such as creation of a pair of black holes, which change a universe that is already present. According to the simple interpretation of Euclidean solutions in Einstein's theory, the most probable path to black hole creation is discontinuous via nothing as an intermediate state. In the present paper we have considered several modifications of Einstein's theory that allow continuous histories as saddle points of the Euclidean action between two finite universes. Considered as a matter source, these modifications involve extreme forms of the stress-energy tensor because the Ricci tensor will typically have at least one negative eigenvalue. Therefore the formation of baby universes is a possible competing process.

A matter field that can form sufficiently small domain walls is a universal connector, replacing the intermediate state of nothing with at least a small three-sphere. Higher-order Lagrangians in the scalar curvature have to be fine tuned to allow the desired continuous histories. In many ways the most successful solution involves a scalar C-field of negative (but small) coupling constant.

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